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A Self-Similar Piston Problem

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SUMMARY

Self-similar solutions for the flow between a piston and the strong shock propagating non-uniformly into an ideal gas at rest obeying the power law density distribution are investigated. The conditions for the existency of the solutions are derived. Numerical solutions are obtained.

1. Introduction

Numerical solutions for the similarity flows of an ideal gas driven out by spherical piston expanding with uniform velocity were first investigated, independently, by Sedov [1] and Taylor [2]. This problem is extended by Krasheninikova [3] to the case when the piston expands non-uniformly with a velocity U given by

$$U = U_0 t^n \qquad (n > -1)\,, \tag{1}$$

where U_0 is a constant. The solutions of this problem are analogous to that of the solutions of hypersonic flow past the power law bodies obtained by Lees and Kubota [4] who have shown that the condition for the existency of the solution is $-\nu/(\nu+2) < n \leqq 0$ where $\nu = 1,2,3$ for plane, cylindrical and spherical flows respectively. The same condition is also derived by Grigorin [5]. In all these works the gas ahead of the shock is assumed to be uniform and at rest. Recently Helliwell [6] studied the piston problem, in which the piston velocity is assumed to be of the form (1) and the density of a gas ahead of the shock is satisfying the law

$$\rho_1 = A r^{-w} \qquad (w > 0)\,, \tag{2}$$

where A is a constant. This problem has been considered as a particular case of a radiative piston problem in which there exists, by dimensional considerations, a relation between n and w, namely, $n = -w/(w+5)$. It appears that this relation is taken for granted in the case of non-radiative piston problem also and the whole analysis is based upon this relation.

In this paper we consider the self-similar piston problem in which the piston velocity is given by (1). The shock is assumed to be strong and propagating in a medium at rest in which the density obeys equation (2). This problem with spherical symmetry has got importance in astrophysics (See Parker [7]). We first note that there are only two independent dimensional constants U_0 and A involved in the problem and so the self-similarity exists (See Sedov [8]). Also there can not be in general any explicit relation between n and w as given by Helliwell. On the other hand it is shown that for all physically meaningful flows the ranges for n and w are

$$-(\nu - w)/(\nu + 2 - w) < n \leqq -w(\gamma - 1)/[w(\gamma - 1) + 2] \quad \text{and} \quad 0 < w < \nu/\gamma\,,$$

where γ is the ratio of specific heats. These conditions on n ensure the finiteness of density and pressure drag on the piston surface. Also it is shown that when $n = -w(\gamma-1)/[w(\gamma-1)+2]$, the flow becomes homentropic. The problem with $\gamma = \frac{7}{5}$, $n = -w/(w+5)$ considered by Helliwell [6] corresponds to homentropic flow. Numerical solutions for $\nu = 3$, $\gamma = \frac{7}{5}$ and $w = 1.5$ are given using the Adams-Moulton method.

2. Basic Equations

The equation of motion, continuity and energy for one-dimensional unsteady flow of a perfect gas can be written in the form (see Sedov [8])

$$\lambda(V-\delta)V'+\lambda\frac{P'}{R}+V(V-1)-(w-2)\frac{P}{R}=0\,, \tag{3}$$

$$\lambda\left[V'+(V-\delta)\frac{R'}{R}\right]+(v-w)V=0\,, \tag{4}$$

$$\lambda(V-\delta)\left[\frac{P'}{P}-\gamma\frac{R'}{R}\right]-2+[w(\gamma-1)+2]V=0 \tag{5}$$

by the following transformation

$$v=\frac{r}{t}v(\lambda)\,,\quad \rho=\frac{A}{r^{w}}R(\lambda)\,,\quad p=\frac{A}{r^{w-2}t^{2}}P(\lambda)\,, \tag{6}$$

$$\lambda=\left(\frac{\delta\bar{\lambda}}{U_0}\right)rt^{-\delta}\,,\quad \delta=1+n\,. \tag{7}$$

The similarity variable λ is taken in the form (7) by considering U_0 and A as the basic dimensional constants involved in the problem and it takes the values $\bar{\lambda}$ and 1 at the piston surface and behind the shock respectively. The rest of the symbols have got their usual meaning.

From the equations (3)–(5) one can obtain

$$\lambda\frac{dV}{d\lambda}=\frac{V(V-1)(V-\delta)+(K-vV)z}{z-(V-\delta)^2}\,, \tag{8}$$

$$\frac{dz}{dV}=\frac{z\{[2(V-1)+v(\gamma-1)V](V-\delta)^2-(\gamma-1)V(V-1)(V-\delta)-z[2(V-1)+K(\gamma-1)]\}}{(V-\delta)[V(V-1)(V-\delta)+(K-vV)z]}\,, \tag{9}$$

where

$$z=\gamma\frac{P}{R}\,,\quad \gamma K=(w-2)\delta+2\,. \tag{10}$$

The strong shock conditions are given by

$$V(1)=\frac{2\delta}{\gamma+1}\,,\quad R(1)=\frac{\gamma+1}{\gamma-1}\,,\quad P(1)=\frac{2\delta^2}{\gamma+1}\,,\quad z(1)=\frac{2\gamma(\gamma-1)\delta^2}{(\gamma+1)^2} \tag{11}$$

and the kinematic condition on the piston gives

$$V(\bar{\lambda})=\delta\,. \tag{12}$$

The region of interest in V–z plane is

$$z>0\,,\quad z-(V-\delta)^2>0\,,\quad \frac{2\delta}{\gamma+1}\leqq V\leqq\delta\,. \tag{13}$$

3. Conditions for Existence of Solutions

The total energy of the flow between piston surface and the shock front can be written, using (6) and (7), as

$$E=A\varepsilon_v\left(\frac{U_0}{\delta\bar{\lambda}}\right)^{v+2-w}t^{\delta(v+2-w)-2}\int_{\bar{\lambda}}^{1}\left[\tfrac{1}{2}RV^2+\frac{P}{\gamma-1}\right]\lambda^{v+1-w}d\lambda\,, \tag{14}$$

where $\varepsilon_\nu = 2^{\nu-1}\pi^{\frac{1}{2}(\nu-1)(4-\nu)}$. For the flows driven out by the piston, the energy always increases with time. This is possible only if

$$n > -\left[\frac{\nu - w}{\nu + 2 - w}\right], \qquad w < \nu . \tag{15}$$

The second condition of (15) is required to make sure that for all physically meaningful solutions δ must lie between zero and one. These conditions ensure that the pressure drag on the piston is finite. Further it is necessary that $dV/d\lambda < 0$ in the domain of interest. So it follows from (8), (10) and (13) that a physically meaningful solutions does not exist if $K - \nu\delta \geqq 0$ i.e.,

$$\delta \leqq \frac{2}{\nu\gamma + 2 - w} . \tag{16}$$

From the equations (4) and (5) one can get the following integral

$$z = CR^{\{[\nu(\gamma-1)+2]\delta-2\}/(\nu-w)\delta}(V-\delta)^{\{[w(\gamma-1)+2]\delta-2\}/(\nu-w)\delta}\lambda^{-2/\delta} , \tag{17}$$

where C is a constant of integration to be determined from the conditions (11). From this integral it is clear that $R \to (V-\delta)^m$ as $\lambda \to \bar{\lambda}$, where

$$m = \frac{2 - [w(\gamma-1)+2]\delta}{(\gamma\nu + 2 - w)\delta - 2} . \tag{18}$$

Thus it follows from (16) and (18) that the density at the piston surface is finite if

$$\frac{2}{\nu\gamma + 2 - w} < \delta \leqq \frac{2}{w(\gamma-1)+2} . \tag{19}$$

For all gases (with $\gamma > 1$) the ranges for n and w can be obtained from (15) and (19) as

$$-\frac{(\nu - w)}{\nu + 2 - w} < n \leqq -\frac{w(\gamma-1)}{w(\gamma-1)+2}, \qquad 0 < w < \frac{\nu}{\gamma} . \tag{20}$$

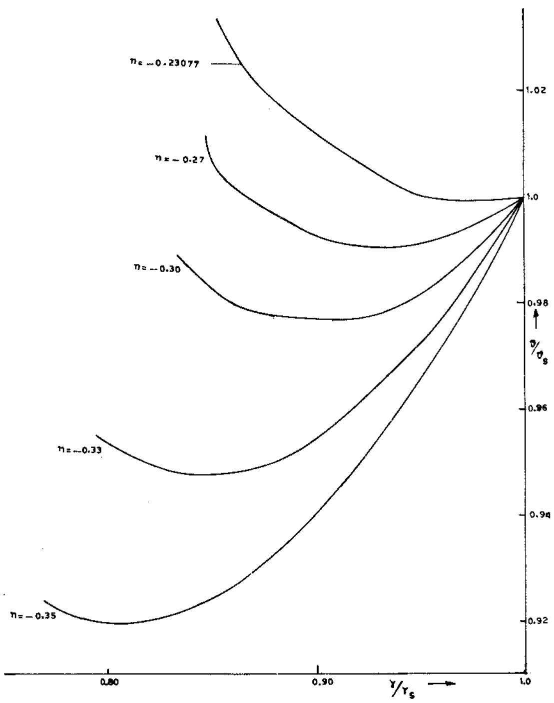

Figure 1. Velocity distribution.

These are the conditions on n and w for the existency of physically meaningful solutions. The conditions for existency of the solutions derived by Lees and Kubota [4] follow from (20) when $w=0$. When n attains the maximum value i.e., $n=-w(\gamma-1)/[w(\gamma-1)+2]$ the density on the piston attains a constant value and the flow is homentropic. Incidently, we note that the case $\gamma=\frac{7}{5}$ and $n=-w/(w+5)$, considered by Helliwell, corresponds to the homentropic flow problem.

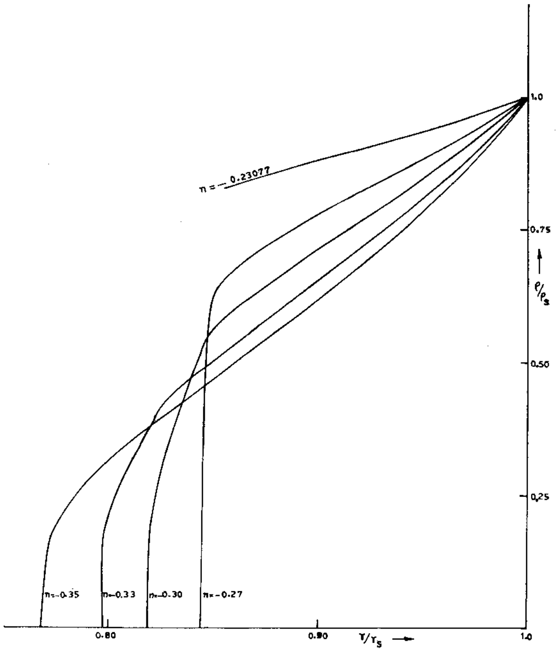

Figure 2. Density distribution.

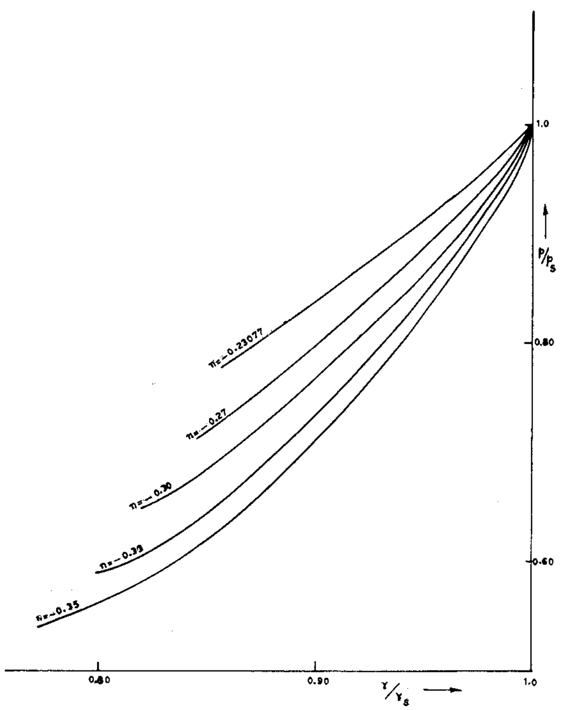

Figure 3. Pressure distribution.

4. Solutions

Once the equation (9) is solved for z in terms of V then $V(\lambda)$, $R(\lambda)$ and $P(\lambda)$ can be found from the equations (8), (4) and (17) respectively. Analytical solutions of (9), a nonlinear equation of Poincaré's type, seem to be not obtainable with conditions (20). But a qualitative description of the integral curves in V–z plane can be given by finding out the singularities of (9) and their nature. Here we note that some of the singularities of (9), for example $V=K/v$, $z=\infty$, depend upon v, γ, w and n while Helliwell's work shows that they are independent of w. The reason for this lies in the relation $n=-w/(w+5)$ taken by Helliwell. But for clear understanding of the field variables describing the flow between the piston and the shock it is easy to integrate numerically the equations (3), (4) and (5) starting with the known values at the shock given by (11) and continuing until the value $\bar{\lambda}$ is reached such that $V(\bar{\lambda})=\delta$. Since the case $v=3$ is of physical importance, we have given the numerical solutions for $\gamma=\frac{7}{5}$, $w=1.5$ for different values of n in Figs. (1), (2) and (3). Here the suffix s denotes the values behind the shock front and $\lambda=r/r_s$. The expressions v/v_s, ρ/ρ_s and p/p_s can be calculated from V, R and P respectively with the help of (6) and (11). For example $v/v_s=(\gamma+1)\lambda V/2\delta$. Homentropic flow corresponds to $n=-0.23077$. The computation was carried out on the CDC-3600. The method used is the Adams-Moutlon method using Runga-Kutta starter and partial Double Precision Arithmetic.

In conclusion we wish to thank the referee for his comments.

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